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Multi-step implicit iterative methods with regularization for minimization problems and fixed point problems

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Abstract

In this paper we introduce a multi-step implicit iterative scheme with regularization for finding a common solution of the minimization problem (MP) for a convex and continuously Fréchet differentiable functional and the common fixed point problem of an infinite family of nonexpansive mappings in the setting of Hilbert spaces. The multi-step implicit iterative method with regularization is based on three well-known methods: the extragradient method, approximate proximal method and gradient projection algorithm with regularization. We derive a weak convergence theorem for the sequences generated by the proposed scheme. On the other hand, we also establish a strong convergence result via an implicit hybrid method with regularization for solving these two problems. This implicit hybrid method with regularization is based on the CQ method, extragradient method and gradient projection algorithm with regularization.

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Keywords: multi-step implicit iterative method with regularization; implicit hybrid method with regularization; minimization problem; nonexpansive mapping; inverse-strong monotonicity; Opial's condition; Kadec-Klee property

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let $S: C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A: C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$. In particular, if $L = 1$, then A is called a nonexpansive mapping [1]; if $L \in [0,1)$ then A is called a contraction.

Let $f: C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the minimization problem (MP) of minimizing f over the constraint set C

$$\min_{x\in C} f(x). \tag{1.1}$$

We denote by Γ the set of minimizers of MP (1.1) which are assumed to be nonempty.

On the other hand, consider the following variational inequality problem (VIP): find a $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of VIP (1.2) is denoted by $\mathrm{VI}(C, A)$.

We remark that VIP (1.2) was first discussed by Lions [2] and now is well known. There are a lot of different approaches towards solving VIP (1.2) in finite-dimensional and infinite-dimensional spaces, and the research is intensively investigated. VIP (1.2) has many applications in computational mathematics, mathematical physics, operations research, mathematical economics, optimization theory, and other fields; see, *e.g.*, [3–6] and the references therein.

Recently, motivated by the work of Takahashi and Zembayashi [7], Cholamjiak [8] introduced a new hybrid projection algorithm for finding a common element of the set of solutions of the equilibrium problem and the set of solutions of the variational inequality problem and the set of fixed points of relatively quasi-nonexpansive mappings in a Banach space. Here, the involved operator in [8] is an inverse-strongly monotone operator. Furthermore, Nadezhkina and Takahashi [9] introduced an iterative process for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ and obtained a strong convergence theorem.

Theorem NT (see [9, Theorem 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and L-Lipschitz-continuous mapping and $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{\mathrm{Fix}(S) \cap \mathrm{VI}(C,A)} x$.

Also, it is remarkable that the joint work of Nadezhkina and Takahashi [9], which introduced a new iterative method, combines Korpelevich's extragradient method and the so-called CQ method. We note that Nadezhkina and Takahashi employed the monotonicity and Lipschitz-continuity of A to define a maximal monotone operator T [10]. However, if the mapping A is pseudomonotone Lipschitz-continuous, then T is not necessarily a maximal monotone operator. To overcome this difficulty, Ceng *et al.* [11] suggested another iterative method. They established necessary and sufficient mild conditions such that the sequences generated by their proposed method converge weakly to some common solution of VIP (1.2) and the common fixed point problem of a finite family of nonexpansive mappings.

Theorem CTY ([11, Theorem 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a pseudomonotone, k-Lipschitz-continuous and (w,s)-sequentially-continuous mapping of C into H, and let $\{S_n\}_{i=1}^N$ be N nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N \operatorname{Fix}(S_i) \cap \operatorname{VI}(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S_n P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ \quad \text{find } x_{n+1} \in C_n \text{ such that } \langle x_n - x_{n+1} + e_n - \sigma_n A x_{n+1}, x_{n+1} - x\rangle \geq -\varepsilon_n,\ \forall x \in C_n, \end{cases}$$

for every $n = 1, 2, \dots$, where $S_n = S_{n \bmod N}$, $\{e_n\}$ is an error sequence in H such that $\sum_{n=1}^\infty \|e_n\| < \infty$ and the following conditions hold:
(i) $\{\sigma_n\} \subset (0, 1/k)$, $\{\varepsilon_n\} \subset [0,\infty)$ *and* $\sum_{n=1}^\infty \varepsilon_n < \infty$;
(ii) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 1/k)$;
(iii) $\{\alpha_n\} \subset [0,c]$ *for some* $c \in [0,1)$.
Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge weakly to the same element of $\bigcap_{i=1}^N \operatorname{Fix}(S_i) \cap \operatorname{VI}(C,A)$ if and only if $\liminf_{n\to\infty} \langle A x_n, x - x_n\rangle \geq 0$, $\forall x \in C$.

In this paper, we aim to find a common solution of the minimization problem (MP) for a convex and continuously Fréchet differentiable functional and the common fixed point problem of an infinite family of nonexpansive mappings in the setting of Hilbert spaces. Motivated and inspired by the research going on in this area, we propose two iterative schemes for this purpose. One is called a multi-step implicit iterative method with regularization which is based on three well-known methods: extragradient method, approximate proximal method and gradient projection algorithm with regularization. Another is an implicit hybrid method with regularization which is based on the CQ method, extragradient method and gradient projection algorithm with regularization. Weak and strong convergence results for these two schemes are established, respectively. Recent results in this direction can be found, *e.g.*, in [7–32].

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1 *For given* $x \in H$ *and* $z \in C$:
(i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0$, $\forall y \in C$;
(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;
(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2$, $\forall y \in H$.

Consequently, P_C is nonexpansive and monotone.

Definition 2.1 A mapping $A : C \to H$ is said to be:
(a) pseudomonotone if for all $x, y \in C$

$$\langle Ay - Ay, x - y \rangle \geq 0 \quad \Rightarrow \quad \langle Ax, x - y \rangle \geq 0;$$

(b) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(c) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C;$$

(d) α-inverse-strongly monotone (α-ism) if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Recall that a mapping $S : C \to C$ is said to be nonexpansive [1] if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Denote by $\mathrm{Fix}(S)$ the set of fixed points of S; that is, $\mathrm{Fix}(S) = \{x \in C : Sx = x\}$. It can be easily seen that if $S : C \to C$ is nonexpansive, then $I - S$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

We need some facts and tools which are listed as lemmas below.

Lemma 2.1 *Let* X *be a real inner product space. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in X.$$

Lemma 2.2 *Let* $\{x_n\}$ *be a bounded sequence in a reflexive Banach space* X. *If* $\omega_w(\{x_n\}) = \{x\}$, *then* $x_n \rightharpoonup x$.

Lemma 2.3 *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem, the characterization of the projection* (*see Proposition* 2.1(i)) *implies*

$$u \in \mathrm{VI}(C,A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \forall \lambda > 0.$$

Lemma 2.4 *Let H be a real Hilbert space. Then the following hold*:

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0,1]$ *with* $\lambda + \mu = 1$;

(c) *If* $\{x_n\}$ *is a sequence in H such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x-y\|^2, \quad \forall y \in H.$$

Lemma 2.5 ([33, Lemma 2.5]) *Let H be a real Hilbert space. Given a nonempty closed convex subset of H and points $x, y, z \in H$ and given also a real number $a \in \mathbf{R}$, the set*

$$\left\{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\right\}$$

is convex (*and closed*).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself and let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence in $[0,1]$. For any $n \ge 1$, define a mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \xi_n S_n U_{n,n+1} + (1-\xi_n)I, \\ U_{n,n-1} = \xi_{n-1} S_{n-1} U_{n,n} + (1-\xi_{n-1})I, \\ \cdots, \\ U_{n,k} = \xi_k S_k U_{n,k+1} + (1-\xi_k)I, \\ U_{n,k-1} = \xi_{k-1} S_{k-1} U_{n,k} + (1-xi_{k-1})I, \\ \cdots, \\ U_{n,2} = \xi_2 S_2 U_{n,3} + (1-\xi_2)I, \\ W_n = U_{n,1} = \xi_1 S_1 U_{n,2} + (1-\xi_1)I. \end{cases} \tag{2.1}$$

Such W_n is called a W-mapping generated by $\{S_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$. We need the following lemmas for proving our main results.

Lemma 2.6 [34] *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} S_n$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le b < 1$ for all $i \ge 1$. Then, for every $x \in C$ and $k \ge 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.6, one can define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C.$$

Lemma 2.7 ([34]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S_1, S_2, \dots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n)$ is nonempty, and let $\xi_1, \xi_2, \dots$ be real numbers such that $0 < \xi_i \le b < 1$ for all $i \ge 1$. Then*

$$\operatorname{Fix}(W) = \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n).$$

Lemma 2.8 ([35]) *If $\{x_n\}$ is a bounded sequence in C, then*

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0.$$

Lemma 2.9 ([36, Demiclosedness principle]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping such that* $\operatorname{Fix}(S) \neq \emptyset$. *Then S is demiclosed on C, i.e., if $y_n \rightharpoonup z \in C$ and $y_n - Sy_n \to y$, then $(I - S)z = y$.*

To prove a weak convergence theorem by the multi-step implicit iterative method with regularization for MP (1.1) and infinitely many nonexpansive mappings $\{S_n\}_{n=1}^{\infty}$, we need the following lemma due to Osilike *et al.* [37].

Lemma 2.10 ([37, p.80]) *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n) a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Corollary 2.1 ([38, p.303]) *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.11 ([36]) *Every Hilbert space H has the Kadec-Klee property; that is, given a sequence $\{x_n\} \subset H$ and a point $x \in H$, we have*

$$\left.\begin{array}{r} \|x_n\| \to \|x\| \\ x_n \rightharpoonup x \end{array}\right\} \quad \Longrightarrow \quad x_n \to x.$$

It is well known that every Hilbert space H satisfies Opial's condition [39], *i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y,g) \in G(T)$ implies $f \in Tx$. Let $A : C \to H$ be a monotone, L-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

It is known that in this case T is maximal monotone, and $0 \in Tv$ if and only if $v \in \Omega$; see [10].

3 Weak convergence theorem

In this section, we derive weak convergence criteria for a multi-step implicit iterative method with regularization for finding a common solution of the common fixed point problem of infinitely many nonexpansive mappings $\{S_n\}_{n=1}^\infty$ and MP (1.1) for a convex functional $f : C \to \mathbf{R}$ with an L-Lipschitz continuous gradient ∇f. This implicit iterative method with regularization is based on the extragradient method, approximate proximal method and gradient projection algorithm (GPA) with regularization.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let W_n be a W-mapping defined by* (2.1), *let $\nabla f : C \to H$ be an L-Lipschitz continuous mapping with $L > 0$, and let $\{S_n\}_{n=1}^\infty$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty \operatorname{Fix}(S_n) \cap \Gamma$ is nonempty and bounded. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \mu_n \nabla f_{\alpha_n}(x_n) - \lambda_n(1 - \mu_n)\nabla f_{\alpha_n}(y_n)), \\ t_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - \lambda_n(1 - \mu_n)\nabla f_{\alpha_n}(t_n)), \\ z_n = \gamma_n x_n + (1 - \gamma_n) W_n t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n}((1 - \beta_n)x_n + e_n - \sigma_n \nabla f(x_{n+1})), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{e_n\} \subset H$ an error sequence with $\sum_{n=1}^\infty \|e_n\| < \infty$, and $\theta_n = \alpha_n \frac{18}{\mu_n^4} \Delta_n$ with

$$\Delta_n = \sup\left\{ \|x_n - z\|^2 + \left(1 + \frac{1}{18L^2}\right)\|z\|^2 : z \in \bigcap_{n=1}^\infty \operatorname{Fix}(S_n) \cap \Gamma \right\} < \infty.$$

Assume the following conditions hold:

(i) $\{\alpha_n\} \subset (0,\infty)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\{\gamma_n\} \subset [0,c]$ *for some* $c \in [0,1)$;
(iii) $\{\mu_n\} \subset (0,1]$ *and* $\lim_{n\to\infty} \mu_n = 1$;
(iv) $\lambda_n(\alpha_n + L) < 1$, $\forall n \geq 1$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0,1/L)$;
(v) $\{\sigma_n\} \subset (0,1/L)$ *and* $\{\beta_n\} \subset [0,1]$ *satisfy* $\sum_{n=1}^\infty \beta_n < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *generated by* (3.1) *converge weakly to some* $u \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$.

Remark 3.1 In the proof of Theorem 3.1 below, we show that every C_n is closed and convex and that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma \subset C_n$, $\forall n \geq 1$.

Now, we observe that for all $x, y \in C$ and all $n \geq 1$,

$$\begin{aligned}
&\bigl\|P_C\bigl(x_n - \lambda_n\mu_n\nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(x)\bigr) \\
&\qquad - P_C\bigl(x_n - \lambda_n\mu_n\nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y)\bigr)\bigr\| \\
&\quad \leq \bigl\|\bigl(x_n - \lambda_n\mu_n\nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(x)\bigr) \\
&\qquad - \bigl(x_n - \lambda_n\mu_n\nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y)\bigr)\bigr\| \\
&\quad = \lambda_n(1-\mu_n)\bigl\|\nabla f_{\alpha_n}(x) - \nabla f_{\alpha_n}(y)\bigr\| \\
&\quad \leq \lambda_n(\alpha_n + L)\|x - y\|.
\end{aligned}$$

Hence, by the Banach contraction principle, we know that for each $n \geq 1$ there exists a unique $y_n \in C$ such that

$$y_n = P_C\bigl(x_n - \lambda_n\mu_n\nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y_n)\bigr). \tag{3.2}$$

Also, observe that for all $x, y \in C$ and all $n \geq 1$,

$$\begin{aligned}
&\bigl\|P_C\bigl(x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(x)\bigr) \\
&\qquad - P_C\bigl(x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y)\bigr)\bigr\| \\
&\quad \leq \bigl\|\bigl(x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(x)\bigr) \\
&\qquad - \bigl(x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y)\bigr)\bigr\| \\
&\quad = \lambda_n(1-\mu_n)\bigl\|\nabla f_{\alpha_n}(x) - \nabla f_{\alpha_n}(y)\bigr\| \\
&\quad \leq \lambda_n(\alpha_n + L)\|x - y\|.
\end{aligned}$$

So, by the Banach contraction principle, we know that for each $n \geq 1$ there exists a unique $z_n \in C$ such that

$$t_n = P_C\bigl(x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)\bigr). \tag{3.3}$$

In addition, observe that for all $x, y \in C$ and all $n \geq 1$,

$$\begin{aligned}
&\bigl\|P_{C_n}\bigl((1-\beta_n)x_n + e_n - \sigma_n\nabla f(x)\bigr) - P_{C_n}\bigl((1-\beta_n)x_n + e_n - \sigma_n\nabla f(y)\bigr)\bigr\| \\
&\quad \leq \bigl\|\bigl((1-\beta_n)x_n + e_n - \sigma_n\nabla f(x)\bigr) - \bigl((1-\beta_n)x_n + e_n - \sigma_n\nabla f(y)\bigr)\bigr\| \\
&\quad = \sigma_n\bigl\|\nabla f(x) - \nabla f(y)\bigr\| \\
&\quad \leq \sigma_n L\|x - y\|.
\end{aligned}$$

Thus, by the Banach contraction principle, we know that for each $n \geq 1$ there exists a unique $x_{n+1} \in C_n$ such that

$$x_{n+1} = P_{C_n}\big((1-\beta_n)x_n + e_n - \sigma_n \nabla f(x_{n+1})\big). \tag{3.4}$$

Therefore, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (3.1) are well defined.

Next, we divide our detailed proof into several propositions. For this purpose, in the sequel, we assume that all our assumptions are satisfied.

Proposition 3.1 $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma \subset C_n$, $\forall n \geq 1$.

Proof First we note that every set C_n is closed and convex. As a matter of fact, since the defining inequality in C_n is equivalent to the inequality

$$\big\langle 2(x_n - z_n), z\big\rangle \leq \|x_n\|^2 - \|z_n\|^2 + \theta_n,$$

by Lemma 2.5 we also have that C_n is convex and closed for every $n = 1, 2, \ldots$. Also, note that the L-Lipschitz continuity of the gradient ∇f implies that ∇f is $1/L$-ism [31], that is,

$$\big\langle \nabla f(x) - \nabla f(y), x - y\big\rangle \geq \frac{1}{L}\big\|\nabla f(x) - \nabla f(y)\big\|^2, \quad \forall x, y \in C.$$

Observe that

$$\begin{aligned}
&(\alpha + L)\big\langle \nabla f_\alpha(x) - \nabla f_\alpha(y), x - y\big\rangle \\
&\quad = (\alpha + L)\big[\alpha\|x - y\|^2 + \big\langle \nabla f(x) - \nabla f(y), x - y\big\rangle\big] \\
&\quad = \alpha^2\|x - y\|^2 + \alpha\big\langle \nabla f(x) - \nabla f(y), x - y\big\rangle + \alpha L\|x - y\|^2 \\
&\qquad + L\big\langle \nabla f(x) - \nabla f(y), x - y\big\rangle \\
&\quad \geq \alpha^2\|x - y\|^2 + 2\alpha\big\langle \nabla f(x) - \nabla f(y), x - y\big\rangle + \big\|\nabla f(x) - \nabla f(y)\big\|^2 \\
&\quad = \big\|\alpha(x - y) + \nabla f(x) - \nabla f(y)\big\|^2 \\
&\quad = \big\|\nabla f_\alpha(x) - \nabla f_\alpha(y)\big\|^2.
\end{aligned}$$

Hence, it follows that $\nabla f_\alpha = \alpha I + \nabla f$ is $1/(\alpha + L)$-ism. Now, take $u \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$ arbitrarily. Taking into account $\lambda_n(\alpha_n + L) < 1$, $\forall n \geq 1$, we deduce that

$$\begin{aligned}
&\big\|x_n - u - \lambda_n\mu_n\big(\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(u)\big)\big\|^2 \\
&\quad \leq \|x_n - u\|^2 + \lambda_n\mu_n\left(\lambda_n\mu_n - \frac{2}{\alpha_n + L}\right)\big\|\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(u)\big\|^2 \\
&\quad \leq \|x_n - u\|^2 + \lambda_n\left(\lambda_n - \frac{2}{\alpha_n + L}\right)\big\|\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(u)\big\|^2 \\
&\quad \leq \|x_n - u\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - u\| &= \big\| P_C\big(x_n - \lambda_n\mu_n \nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y_n)\big) \\
&\quad - P_C\big(u - \lambda_n\mu_n\nabla f(u) - \lambda_n(1-\mu_n)\nabla f(u)\big)\big\| \\
&\le \big\| \big(x_n - \lambda_n\mu_n \nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y_n)\big) \\
&\quad - \big(u - \lambda_n\mu_n\nabla f(u) - \lambda_n(1-\mu_n)\nabla f(u)\big)\big\| \\
&\le \big\| x_n - u - \lambda_n\mu_n\big(\nabla f_{\alpha_n}(x_n) - \nabla f(u)\big)\big\| + \lambda_n(1-\mu_n)\big\|\nabla f_{\alpha_n}(y_n) - \nabla f(u)\big\| \\
&\le \big\| x_n - u - \lambda_n\mu_n\big(\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(u)\big)\big\| + \lambda_n\mu_n\big\|\nabla f_{\alpha_n}(u) - \nabla f(u)\big\| \\
&\quad + \lambda_n(1-\mu_n)\big[\big\|\nabla f_{\alpha_n}(y_n) - \nabla f_{\alpha_n}(u)\big\| + \big\|\nabla f_{\alpha_n}(u) - \nabla f(u)\big\|\big] \\
&\le \|x_n - u\| + \lambda_n\mu_n\alpha_n\|u\| + \lambda_n(1-\mu_n)\big[(\alpha_n + L)\|y_n - u\| + \alpha_n\|u\|\big] \\
&= \|x_n - u\| + \lambda_n(1-\mu_n)(\alpha_n + L)\|y_n - u\| + \lambda_n\alpha_n\|u\| \\
&\le \|x_n - u\| + (1-\mu_n)\|y_n - u\| + \lambda_n\alpha_n\|u\|,
\end{aligned}$$

which implies that

$$\|y_n - u\| \le \frac{1}{\mu_n}\big(\|x_n - u\| + \lambda_n\alpha_n\|u\|\big). \tag{3.5}$$

Meantime, we also have

$$\begin{aligned}
&\|t_n - u\| \\
&\quad = \big\| P_C\big(x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)\big) \\
&\qquad - P_C\big(u - \lambda_n\nabla f(u) - \lambda_n(1-\mu_n)\nabla f(u)\big)\big\| \\
&\quad \le \big\| \big(x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)\big) - \big(u - \lambda_n\nabla f(u) - \lambda_n(1-\mu_n)\nabla f(u)\big)\big\| \\
&\quad \le \|x_n - u\| + \lambda_n\big\|\nabla f_{\alpha_n}(y_n) - \nabla f(u)\big\| + \lambda_n(1-\mu_n)\big\|\nabla f_{\alpha_n}(t_n) - \nabla f(u)\big\| \\
&\quad \le \|x_n - u\| + \lambda_n\big[\big\|\nabla f_{\alpha_n}(y_n) - \nabla f_{\alpha_n}(u)\big\| + \big\|\nabla f_{\alpha_n}(u) - \nabla f(u)\big\|\big] \\
&\qquad + \lambda_n(1-\mu_n)\big[\big\|\nabla f_{\alpha_n}(t_n) - \nabla f_{\alpha_n}(u)\big\| + \big\|\nabla f_{\alpha_n}(u) - \nabla f(u)\big\|\big] \\
&\quad \le \|x_n - u\| + \lambda_n\big[(\alpha_n + L)\|y_n - u\| + \alpha_n\|u\|\big] + \lambda_n(1-\mu_n)\big[(\alpha_n + L)\|t_n - u\| + \alpha_n\|u\|\big] \\
&\quad \le \|x_n - u\| + \|y_n - u\| + \lambda_n\alpha_n\|u\| + (1-\mu_n)\|t_n - u\| + \lambda_n(1-\mu_n)\alpha_n\|u\| \\
&\quad \le \|x_n - u\| + \|y_n - u\| + (2-\mu_n)\lambda_n\alpha_n\|u\| + (1-\mu_n)\|t_n - u\|,
\end{aligned}$$

which hence implies that

$$\begin{aligned}
\|t_n - u\| &\le \frac{1}{\mu_n}\big[\|x_n - u\| + \|y_n - u\| + (2-\mu_n)\lambda_n\alpha_n\|u\|\big] \\
&\le \frac{1}{\mu_n}\left\{\|x_n - u\| + \frac{1}{\mu_n}\big(\|x_n - u\| + \lambda_n\alpha_n\|u\|\big) + (2-\mu_n)\lambda_n\alpha_n\|u\|\right\} \\
&= \left(\frac{1}{\mu_n} + \frac{1}{\mu_n^2}\right)\|x_n - u\| + \left(\frac{1}{\mu_n^2} + \frac{2-\mu_n}{\mu_n}\right)\lambda_n\alpha_n\|u\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1+2\mu_n-\mu_n^2}{\mu_n^2}\bigl(\|x_n-u\|+\lambda_n\alpha_n\|u\|\bigr) \\
&\leq \frac{1+2\mu_n}{\mu_n^2}\bigl(\|x_n-u\|+\lambda_n\alpha_n\|u\|\bigr). \qquad (3.6)
\end{aligned}$$

Thus, from (3.5) and (3.6) it follows that

$$\begin{aligned}
&\|y_n-u\|+(1-\mu_n)\|t_n-u\| \\
&\quad\leq \frac{1}{\mu_n}\bigl(\|x_n-u\|+\lambda_n\alpha_n\|u\|\bigr)+(1-\mu_n)\frac{1+2\mu_n}{\mu_n^2}\bigl(\|x_n-u\|+\lambda_n\alpha_n\|u\|\bigr) \\
&\quad= \frac{1+2\mu_n(1-\mu_n)}{\mu_n^2}\bigl(\|x_n-u\|+\lambda_n\alpha_n\|u\|\bigr) \\
&\quad\leq \frac{3}{\mu_n^2}\bigl(\|x_n-u\|+\lambda_n\alpha_n\|u\|\bigr),
\end{aligned}$$

which together with $\lambda_n(\alpha_n+L)<1$ implies that

$$\begin{aligned}
\bigl[\|y_n-u\|+(1-\mu_n)\|t_n-u\|\bigr]^2 &\leq \frac{9}{\mu_n^4}\bigl(\|x_n-u\|+\lambda_n\alpha_n\|u\|\bigr)^2 \\
&\leq \frac{9}{\mu_n^4}\bigl(2\|x_n-u\|^2+2\lambda_n^2\alpha_n^2\|u\|^2\bigr) \\
&\leq \frac{9}{\mu_n^4}\bigl(2\|x_n-u\|^2+2\|u\|^2\bigr) \\
&= \frac{18}{\mu_n^4}\|x_n-u\|^2+\frac{18}{\mu_n^4}\|u\|^2. \qquad (3.7)
\end{aligned}$$

Furthermore, from Proposition 2.1(ii), the monotonicity of ∇f, and $u\in\Gamma$, we have

$$\begin{aligned}
\|t_n-u\|^2 &\leq \bigl\|\bigl(x_n-\lambda_n\nabla f_{\alpha_n}(y_n)-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)\bigr)-u\bigr\|^2 \\
&\quad-\bigl\|\bigl(x_n-\lambda_n\nabla f_{\alpha_n}(y_n)-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)\bigr)-t_n\bigr\|^2 \\
&= \bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-u\bigr\|^2 \\
&\quad-\bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-t_n\bigr\|^2+2\lambda_n\bigl\langle\nabla f_{\alpha_n}(y_n),u-t_n\bigr\rangle \\
&= \bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-u\bigr\|^2-\bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-t_n\bigr\|^2 \\
&\quad+2\lambda_n\bigl(\bigl\langle\nabla f_{\alpha_n}(y_n),u-y_n\bigr\rangle+\bigl\langle\nabla f_{\alpha_n}(y_n),y_n-t_n\bigr\rangle\bigr) \\
&= \bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-u\bigr\|^2-\bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-t_n\bigr\|^2 \\
&\quad+2\lambda_n\bigl(\bigl\langle\nabla f_{\alpha_n}(y_n)-\nabla f_{\alpha_n}(u),u-y_n\bigr\rangle+\bigl\langle\nabla f_{\alpha_n}(u),u-y_n\bigr\rangle \\
&\quad+\bigl\langle\nabla f_{\alpha_n}(y_n),y_n-t_n\bigr\rangle\bigr) \\
&\leq \bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-u\bigr\|^2-\bigl\|x_n-\lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)-t_n\bigr\|^2 \\
&\quad+2\lambda_n\bigl(\alpha_n\langle u,u-y_n\rangle+\bigl\langle\nabla f_{\alpha_n}(y_n),y_n-t_n\bigr\rangle\bigr) \\
&= \|x_n-u\|^2-\|x_n-t_n\|^2-2\lambda_n(1-\mu_n)\bigl\langle\nabla f_{\alpha_n}(t_n),t_n-u\bigr\rangle \\
&\quad+2\lambda_n\bigl(\alpha_n\langle u,u-y_n\rangle+\bigl\langle\nabla f_{\alpha_n}(y_n),y_n-t_n\bigr\rangle\bigr)
\end{aligned}$$

$$\begin{aligned}
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\
&\quad + 2\lambda_n\bigl(\alpha_n\langle u, u - y_n\rangle + \langle \nabla f_{\alpha_n}(y_n), y_n - t_n\rangle\bigr) \\
&\quad - 2\lambda_n(1-\mu_n)\bigl(\langle \nabla f_{\alpha_n}(t_n) - \nabla f_{\alpha_n}(u), t_n - u\rangle + \langle \nabla f_{\alpha_n}(u), t_n - u\rangle\bigr) \\
&\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - y_n, t_n - y_n\rangle \\
&\quad + 2\lambda_n\alpha_n\bigl(\langle u, u - y_n\rangle + (1-\mu_n)\langle u, u - t_n\rangle\bigr) \\
&\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - y_n, t_n - y_n\rangle \\
&\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr].
\end{aligned}$$

Since $y_n = P_C(x_n - \lambda_n\mu_n\nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y_n))$ and ∇f_{α_n} $(\alpha_n + L)$-Lipschitz continuous, by Proposition 2.1(i) we have

$$\begin{aligned}
\langle x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - y_n, t_n - y_n\rangle &= \langle x_n - \lambda_n\mu_n\nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y_n) - y_n, t_n - y_n\rangle \\
&\quad + \lambda_n\mu_n\langle \nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(y_n), t_n - y_n\rangle \\
&\le \lambda_n\mu_n\langle \nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(y_n), t_n - y_n\rangle \\
&\le \lambda_n\mu_n\bigl\|\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(y_n)\bigr\|\|t_n - y_n\| \\
&\le \lambda_n(\alpha_n + L)\|x_n - y_n\|\|t_n - y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\|t_n - u\|^2 \\
&\quad\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n(\alpha_n + L)\|x_n - y_n\|\|t_n - y_n\| \\
&\qquad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr] \\
&\quad\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2(\alpha_n + L)^2\|x_n - y_n\|^2 + \|t_n - y_n\|^2 \\
&\qquad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr] \\
&\quad= \|x_n - u\|^2 + \bigl(\lambda_n^2(\alpha_n + L)^2 - 1\bigr)\|x_n - y_n\|^2 + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr] \\
&\quad\le \|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr].
\end{aligned} \tag{3.8}$$

Therefore, from (3.7) and (3.8), together with $z_n = \gamma_n x_n + (1-\gamma_n)W_n t_n$ and $u = W_n u$, by Lemma 2.4(b) we have

$$\begin{aligned}
\|z_n - u\|^2 &= \bigl\|\gamma_n(x_n - u) + (1-\gamma_n)(W_n t_n - u)\bigr\|^2 \\
&\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|W_n t_n - u\|^2 \\
&\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|t_n - u\|^2 \\
&\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\bigl\{\|x_n - u\|^2 + \bigl(\lambda_n^2(\alpha_n + L)^2 - 1\bigr)\|x_n - y_n\|^2 \\
&\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr]\bigr\} \\
&\le \|x_n - u\|^2 + (1-\gamma_n)\bigl(\lambda_n^2(\alpha_n + L)^2 - 1\bigr)\|x_n - y_n\|^2 \\
&\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr]
\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\big] \\
&\le \|x_n - u\|^2 + \alpha_n\big[\lambda_n^2\|u\|^2 + \big(\|y_n - u\| + (1-\mu_n)\|t_n - u\|\big)^2\big] \\
&\le \|x_n - u\|^2 + \alpha_n\left[\frac{1}{L^2}\|u\|^2 + \frac{18}{\mu_n^4}\|x_n - u\|^2 + \frac{18}{\mu_n^4}\|u\|^2\right] \\
&= \|x_n - u\|^2 + \alpha_n\frac{18}{\mu_n^4}\left[\|x_n - u\|^2 + \left(1 + \frac{\mu_n^4}{18L^2}\right)\|u\|^2\right] \\
&\le \|x_n - u\|^2 + \alpha_n\frac{18}{\mu_n^4}\left[\|x_n - u\|^2 + \left(1 + \frac{1}{18L^2}\right)\|u\|^2\right] \\
&\le \|x_n - u\|^2 + \alpha_n\frac{18}{\mu_n^4}\Delta_n \\
&= \|x_n - u\|^2 + \theta_n,
\end{aligned} \tag{3.9}$$

which implies that $u \in C_n$. Therefore,

$$\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_n, \quad \forall n \ge 1,$$

and this completes the proof. □

Proposition 3.2 *The sequences* $\{x_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *are all bounded.*

Proof Since $u \in \Gamma$ and $x_n \in C$ for all $n \ge 1$, from the monotonicity of ∇f, we have

$$\langle \nabla f(u), x_n - u\rangle \ge 0 \quad \text{and} \quad \langle \nabla f(x_n) - \nabla f(u), x_n - u\rangle \ge 0, \quad \forall n \ge 1,$$

which hence implies that

$$\langle \nabla f(x_n), x_n - u\rangle \ge \langle \nabla f(u), x_n - u\rangle \ge 0, \quad \forall n \ge 1. \tag{3.10}$$

Note that $x_{n+1} = P_{C_n}((1-\beta_n)x_n + e_n - \sigma_n\nabla f(x_{n+1}))$ is equivalent to the inequality

$$\langle (1-\beta_n)x_n - x_{n+1} + e_n - \sigma_n\nabla f(x_{n+1}), x_{n+1} - x\rangle \ge 0, \quad \forall x \in C_n.$$

Taking $x = u$ in the last inequality, we deduce

$$\langle (1-\beta_n)x_n - x_{n+1} + e_n - \sigma_n\nabla f(x_{n+1}), x_{n+1} - u\rangle \ge 0,$$

which implies that

$$\langle (1-\beta_n)x_n - x_{n+1} + e_n, x_{n+1} - u\rangle \ge \sigma_n\langle \nabla f(x_{n+1}), x_{n+1} - u\rangle. \tag{3.11}$$

From (3.10) and (3.11) we get

$$\langle (1-\beta_n)x_n - x_{n+1} + e_n, x_{n+1} - u\rangle \ge 0,$$

which can be rewritten as

$$\big\langle (1-\beta_n)(x_n-u)-(x_{n+1}-u)-\beta_n u+e_n, x_{n+1}-u\big\rangle \ge 0. \tag{3.12}$$

It follows that

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \big\langle (1-\beta_n)(x_n-u)-\beta_n u+e_n, x_{n+1}-u\big\rangle \\ &\le (1-\beta_n)\|x_n-u\|\|x_{n+1}-u\|+\beta_n\|u\|\|x_{n+1}-u\|+\|e_n\|\|x_{n+1}-u\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1}-u\| &\le (1-\beta_n)\|x_n-u\|+\beta_n\|u\|+\|e_n\| \\ &\le \max\big\{\|x_n-u\|,\|u\|\big\}+\|e_n\|. \end{aligned} \tag{3.13}$$

By induction, we can obtain

$$\|x_{n+1}-u\| \le \max\big\{\|x_1-u\|,\|u\|\big\}+\sum_{i=1}^{n}\|e_i\|.$$

Since $\sum_{n=1}^{\infty}\|e_n\|<\infty$, we immediately conclude that the sequence $\{x_n\}$ is bounded. Thus, from $\alpha_n\to 0$, $\mu_n\to 1$, $\lambda_n(\alpha_n+L)<1$, (3.5), (3.6) and (3.9) it follows that $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ are bounded. This completes the proof. □

Proposition 3.3 *The following statements hold*:

(i) $\lim_{n\to\infty}\|x_n-u\|$ *exists for each* $u\in\bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n)\cap\Gamma$;

(ii) $\lim_{n\to\infty}\|x_n-x_{n+1}\|=\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|x_n-z_n\|=\lim_{n\to\infty}\|x_n-t_n\|=0$;

(iii) $\lim_{n\to\infty}\|x_n-W_nx_n\|=\lim_{n\to\infty}\|x_n-Wx_n\|=0$.

Proof For each $u\in\bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n)\cap\Gamma$, we get from (3.13)

$$\begin{aligned}\|x_{n+1}-u\| &\le (1-\beta_n)\|x_n-u\|+\beta_n\|u\|+\|e_n\| \\ &\le \|x_n-u\|+\beta_n\|u\|+\|e_n\|.\end{aligned}$$

Since the conditions $\sum_{n=1}^{\infty}\beta_n<\infty$ and $\sum_{n=1}^{\infty}e_n<\infty$ lead to $\sum_{n=1}^{\infty}(\beta_n\|u\|+\|e_n\|)<\infty$, by Corollary 2.1, we know that $\lim_{n\to\infty}\|x_n-u\|$ exists for each $u\in\bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n)\cap\Gamma$. Note that by Lemma 2.4(a) we have from (3.12)

$$\begin{aligned}\|x_n-x_{n+1}\|^2 &= \|x_n-u\|^2-\|x_{n+1}-u\|^2+2\langle x_{n+1}-x_n, x_{n+1}-u\rangle \\ &\le \|x_n-u\|^2-\|x_{n+1}-u\|^2+2\langle -\beta_n x_n+e_n, x_{n+1}-u\rangle \\ &\le \|x_n-u\|^2-\|x_{n+1}-u\|^2+2\big(\beta_n\|x_n\|+\|e_n\|\big)\|x_{n+1}-u\|.\end{aligned}$$

Since $\beta_n\to 0$ and $\|e_n\|\to 0$ as $n\to\infty$, from the existence of $\lim_{n\to\infty}\|x_n-u\|$ and the boundedness of $\{x_n\}$, we obtain that

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0.$$

Since $x_{n+1} \in C_n$, it follows that

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n,$$

which implies that

$$\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n},$$

and hence

$$\begin{aligned}\|x_n - z_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \\ &\le 2\|x_{n+1} - x_n\| + \sqrt{\theta_n} \to 0.\end{aligned}$$

For each $u \in \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \cap \Gamma$, from (3.9) we have

$$\begin{aligned}&(1-\gamma_n)\big(1-\lambda_n^2(\alpha_n+L)^2\big)\|x_n - y_n\|^2 \\ &\quad\le \|x_n - u\|^2 - \|z_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\big] \\ &\quad\le \big(\|x_n - u\| + \|z_n - u\|\big)\|x_n - z_n\| + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + \|t_n - u\|\big].\end{aligned}$$

Since $\{\gamma_n\} \subset [0,c]$, $\{\lambda_n\} \subset [a,b]$, $1 - b^2L^2 > 0$, $\alpha_n \to 0$ and $\|x_n - z_n\| \to 0$ as $n \to \infty$, from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ we conclude that

$$\begin{aligned}&\|x_n - y_n\| \\ &\quad\le \frac{1}{(1-\gamma_n)(1-\lambda_n^2(\alpha_n+L)^2)}\big\{\big(\|x_n - u\| + \|z_n - u\|\big)\|x_n - z_n\| \\ &\qquad + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + \|t_n - u\|\big]\big\} \\ &\quad\le \frac{1}{(1-c)(1-b^2(\alpha_n+L)^2)}\big\{\big(\|x_n - u\| + \|z_n - u\|\big)\|x_n - z_n\| \\ &\qquad + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + \|t_n - u\|\big]\big\} \\ &\quad\to 0.\end{aligned}$$

Utilizing the arguments similar to those in (3.8),

$$\begin{aligned}&\|t_n - u\|^2 \\ &\quad\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n(\alpha_n + L)\|x_n - y_n\|\|t_n - y_n\| \\ &\qquad + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\big] \\ &\quad\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2(\alpha_n+L)^2\|t_n - y_n\|^2 + \|x_n - y_n\|^2 \\ &\qquad + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\big] \\ &\quad= \|x_n - u\|^2 + \big(\lambda_n^2(\alpha_n+L)^2 - 1\big)\|t_n - y_n\|^2 \\ &\qquad + 2\lambda_n\alpha_n\|u\|\big[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\big].\end{aligned}$$

Hence,

$$\begin{aligned}
\|z_n - u\|^2 &\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|t_n - u\|^2 \\
&\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\bigl\{\|x_n - u\|^2 + \bigl(\lambda_n^2(\alpha_n + L)^2 - 1\bigr)\|t_n - y_n\|^2 \\
&\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr]\bigr\} \\
&\le \|x_n - u\|^2 + (1-\gamma_n)\bigl(\lambda_n^2(\alpha_n + L)^2 - 1\bigr)\|t_n - y_n\|^2 \\
&\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + \|t_n - u\|\bigr].
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\gamma_n)\bigl(1 - \lambda_n^2(\alpha_n + L)^2\bigr)\|t_n - y_n\|^2 \\
&\quad \le \|x_n - u\|^2 - \|z_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + \|t_n - u\|\bigr] \\
&\quad \le \bigl(\|x_n - u\| + \|z_n - u\|\bigr)\|x_n - z_n\| + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + \|t_n - u\|\bigr].
\end{aligned}$$

Since $\{\gamma_n\} \subset [0, c]$, $\{\lambda_n\} \subset [a, b]$, $1 - b^2L^2 > 0$, $\alpha_n \to 0$ and $\|x_n - z_n\| \to 0$ as $n \to \infty$, from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ we deduce that

$$\begin{aligned}
&\|t_n - y_n\|^2 \\
&\quad \le \frac{1}{(1-\gamma_n)(1 - \lambda_n^2(\alpha_n + L)^2)}\bigl\{\bigl(\|x_n - u\| + \|z_n - u\|\bigr)\|x_n - z_n\| \\
&\quad\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + \|t_n - u\|\bigr]\bigr\} \\
&\quad \le \frac{1}{(1-c)(1 - b^2(\alpha_n + L)^2)}\bigl\{\bigl(\|x_n - u\| + \|z_n - u\|\bigr)\|x_n - z_n\| \\
&\quad\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + \|t_n - u\|\bigr]\bigr\} \\
&\quad \to 0.
\end{aligned}$$

Taking into consideration that

$$\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|,$$

we also have

$$\lim_{n\to\infty} \|x_n - t_n\| = 0.$$

Since $z_n = \gamma_n x_n + (1-\gamma_n)W_n t_n$, we have

$$(1-\gamma_n)(W_n t_n - t_n) = \gamma_n(t_n - x_n) + z_n - t_n.$$

Then

$$\begin{aligned}
(1-c)\|W_n t_n - t_n\| \le (1-\gamma_n)\|W_n t_n - t_n\| &\le \gamma_n\|t_n - x_n\| + \|z_n - t_n\| \\
&\le (1+\gamma_n)\|t_n - x_n\| + \|z_n - x_n\|,
\end{aligned}$$

and hence $\|t_n - W_n t_n\| \to 0$. Observe also that

$$\begin{aligned}\|x_n - W_n x_n\| &\le \|x_n - t_n\| + \|t_n - W_n t_n\| + \|W_n t_n - W_n x_n\| \\ &\le \|x_n - t_n\| + \|t_n - W_n t_n\| + \|t_n - x_n\| \\ &\le 2\|x_n - t_n\| + \|t_n - W_n t_n\|.\end{aligned}$$

So, we have $\|x_n - W_n x_n\| \to 0$. On the other hand, since $\{x_n\}$ is bounded, from Lemma 2.8, we have $\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0$. Therefore, we obtain

$$\lim_{n\to\infty} \|x_n - W x_n\| = 0,$$

and this completes the proof. □

Proposition 3.4 $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$.

Proof By Proposition 3.3(iii), we know that

$$\lim_{n\to\infty} \|x_n - W x_n\| = 0.$$

Take $\hat{u} \in \omega_w(x_n)$ arbitrarily. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{u}$; hence, we have $\lim_{i\to\infty} \|x_{n_i} - W x_{n_i}\| = 0$. Note that from Lemma 2.9 it follows that $I - W$ is demiclosed at zero. Thus, $\hat{u} \in \mathrm{Fix}(W)$. Now, let us show $\hat{u} \in \Gamma$. Since $x_n - t_n \to 0$ and $x_n - y_n \to 0$, we have $t_{n_i} \rightharpoonup \hat{u}$ and $y_{n_i} \rightharpoonup \hat{u}$. Let

$$Tv = \begin{cases} \nabla f(v) + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, \nabla f)$; see [10] for more details. Let $G(T)$ be the graph of T and let $(v, w) \in G(T)$. Then we have $w \in Tv = \nabla f(v) + N_C v$ and hence $w - \nabla f(v) \in N_C v$. So, we have $\langle v - t, w - \nabla f(v)\rangle \ge 0$ for all $t \in C$. On the other hand, from $t_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - \lambda_n(1 - \mu_n)\nabla f_{\alpha_n}(t_n))$ and $v \in C$, we have

$$\langle x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - \lambda_n(1 - \mu_n)\nabla f_{\alpha_n}(t_n) - t_n, t_n - v\rangle \ge 0$$

and hence

$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + \nabla f_{\alpha_n}(y_n) + (1 - \mu_n)\nabla f_{\alpha_n}(t_n)\right\rangle \ge 0.$$

Therefore, from $\langle v - t, w - \nabla f(v)\rangle \ge 0$ for all $t \in C$ and $t_{n_i} \in C$, we have

$$\begin{aligned}\langle v - t_{n_i}, w\rangle &\ge \langle v - t_{n_i}, \nabla f(v)\rangle \\ &\ge \langle v - t_{n_i}, \nabla f(v)\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f_{\alpha_{n_i}}(y_{n_i}) + (1 - \mu_{n_i})\nabla f_{\alpha_{n_i}}(t_{n_i})\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle v - t_{n_i}, \nabla f(v)\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f(y_{n_i})\right\rangle \\
&\quad - \alpha_{n_i}\langle v - t_{n_i}, y_{n_i}\rangle - (1-\mu_{n_i})\langle v - t_{n_i}, \nabla f_{\alpha_{n_i}}(t_{n_i})\rangle \\
&= \langle v - t_{n_i}, \nabla f(v) - \nabla f(t_{n_i})\rangle + \langle v - t_{n_i}, \nabla f(t_{n_i}) - \nabla f(y_{n_i})\rangle \\
&\quad - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle - \alpha_{n_i}\langle v - t_{n_i}, y_{n_i}\rangle - (1-\mu_{n_i})\langle v - t_{n_i}, \nabla f_{\alpha_{n_i}}(t_{n_i})\rangle \\
&\geq \langle v - t_{n_i}, \nabla f(t_{n_i}) - \nabla f(y_{n_i})\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\quad - \alpha_{n_i}\langle v - t_{n_i}, y_{n_i}\rangle - (1-\mu_{n_i})\langle v - t_{n_i}, \nabla f_{\alpha_{n_i}}(t_{n_i})\rangle.
\end{aligned}
$$

Since $\|\nabla f(t_n) - \nabla f(y_n)\| \to 0$ (due to the Lipschitz continuity of ∇f), $\frac{t_n - x_n}{\lambda_n} \to 0$ (due to $\{\lambda_n\} \subset [a,b]$), $\alpha_n \to 0$ and $\mu_n \to 1$, we obtain $\langle v - \hat{u}, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $\hat{u} \in T^{-1}0$ and hence $\hat{u} \in \mathrm{VI}(C, \nabla f)$. Clearly, $\hat{u} \in \Gamma$. Consequently, $\hat{u} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$. This implies that $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$. □

Finally, according to Propositions 3.1-3.4, we prove the remainder of Theorem 3.1.

Proof It is sufficient to show that $\omega_w(x_n)$ is a single-point set because $x_n - y_n \to 0$ and $x_n - z_n \to 0$ as $n \to \infty$. Since $\omega_w(x_n) \neq \emptyset$, let us take two points $u, \hat{u} \in \omega_w(x_n)$ arbitrarily. Then there exist two subsequences $\{x_{n_j}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup u$ and $x_{m_k} \rightharpoonup \hat{u}$, respectively. In terms of Proposition 3.4, we know that $u, \hat{u} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$. Meantime, according to Proposition 3.3(i), we also know that there exist both $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - \hat{u}\|$. Let us show that $u = \hat{u}$. Assume that $u \neq \hat{u}$. From the Opial condition [39] it follows that

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - u\| &= \liminf_{j\to\infty} \|x_{n_j} - u\| < \liminf_{j\to\infty} \|x_{n_j} - \hat{u}\| \\
&= \lim_{n\to\infty} \|x_n - \hat{u}\| = \liminf_{k\to\infty} \|x_{m_k} - \hat{u}\| \\
&< \liminf_{k\to\infty} \|x_{m_k} - u\| = \lim_{n\to\infty} \|x_n - u\|.
\end{aligned}
$$

This leads to a contradiction. Thus, we must have $u = \hat{u}$. This implies that $\omega_w(x_n)$ is a singleton. Without loss of generality, we may write $\omega_w(x_n) = \{u\}$. Consequently, by Lemma 2.2 we obtain that $x_n \rightharpoonup u \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$. Since $x_n - y_n \to 0$ and $x_n - z_n \to 0$ as $n \to \infty$, we also have that $y_n \rightharpoonup u$ and $z_n \rightharpoonup u$. This completes the proof. □

Remark 3.2 Our Theorem 3.1 improves, extends, supplements and develops Nadezhkina and Takahashi [9, Theorem 3.1] and Ceng *et al.* [11, Theorem 3.1] in the following aspects.

(i) The combination of the problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in [9, Theorem 3.1] and the one of finding an element of $\bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A)$ in [11, Theorem 3.1] is extended to develop the one of finding an element of $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap \Gamma$ in our Theorem 3.1.

(ii) Our Theorem 3.1 drops the required condition $\liminf_{n\to\infty} \langle Ax_n, x - x_n\rangle \geq 0$, $\forall x \in C$ in [11, Theorem 3.1].

(iii) The iterative scheme in [11, Theorem 3.1] is extended to develop the iterative scheme (3.1) of our Theorem 3.1 by virtue of the iterative scheme of [9, Theorem 3.1]. The iterative scheme (3.1) of our Theorem 3.1 is more advantageous and more flexible than the iterative scheme of [11, Theorem 3.1] because it involves several parameter sequences $\{\lambda_n\}$, $\{\mu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$ and $\{e_n\}$.

(iv) The iterative scheme (3.1) in our Theorem 3.1 is very different from every one in [9, Theorem 3.1] and [11, Theorem 3.1] because the final iteration steps of computing x_{n+1} in [9, Theorem 3.1] and [11, Theorem 3.1] are replaced by the implicit iteration step $x_{n+1} = P_{C_n}((1-\beta_n)x_n + e_n - \sigma_n \nabla f(x_{n+1}))$ in the iterative scheme (3.1) of our Theorem 3.1.

(v) The argument technique of our Theorem 3.1 combines the argument one in [9, Theorem 3.1] and the argument one in [11, Theorem 3.1]. Because the problem of finding an element of $\bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i) \cap \varGamma$ in our Theorem 3.1 involves a countable family of nonexpansive mappings $\{S_n\}$, the proof of our Theorem 3.1 depends on the properties of the W-mapping (see Lemmas 2.6-2.8 of Section 2 in this paper). Therefore, the proof of our Theorem 3.1 is very different from every one in [9, Theorem 3.1] and [11, Theorem 3.1].

4 Strong convergence theorem

In this section, we prove a strong convergence theorem via an implicit hybrid method with regularization for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings $\{S_n\}_{n=1}^{\infty}$ and the set of solutions of MP (1.1) for a convex functional $f: C \to \mathbf{R}$ with an L-Lipschitz continuous gradient ∇f. This implicit hybrid method with regularization is based on the CQ method, extragradient method and gradient projection algorithm (GPA) with regularization.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let W_n be a W-mapping defined by* (2.1), *let $\nabla f: C \to H$ be an L-Lipschitz continuous mapping with $L > 0$, and let $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \cap \varGamma$ is nonempty and bounded. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by*

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \mu_n \nabla f_{\alpha_n}(x_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(y_n)), \\ t_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - \lambda_n(1-\mu_n)\nabla f_{\alpha_n}(t_n)), \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \ge 1, \end{cases} \tag{4.1}$$

where $\theta_n = \alpha_n \frac{18}{\mu_n^4} \varDelta_n$ and

$$\varDelta_n = \sup\left\{ \|x_n - z\|^2 + \left(1 + \frac{1}{18L^2}\right)\|z\|^2 : z \in \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \cap \varGamma \right\} < \infty.$$

Assume the following conditions hold:

(i) $\{\alpha_n\} \subset (0,\infty)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\{\gamma_n\} \subset [0, c]$ *for some* $c \in [0, 1)$;
(iii) $\{\mu_n\} \subset (0, 1]$ *and* $\lim_{n\to\infty} \mu_n = 1$;
(iv) $\lambda_n(\alpha_n + L) < 1$, $\forall n \ge 1$ *and* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/L)$.

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *generated by* (4.1) *converge strongly to the same point* $P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(S_n)\cap\Gamma}x$.

Proof Utilizing the condition $\lambda_n(\alpha_n + L) < 1$, $\forall n \ge 1$, and repeating the same arguments as in Remark 3.1, we can see that $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ are defined well. Note that the L-Lipschitz continuity of the gradient ∇f implies that ∇f is $\frac{1}{L}$-ism [31], that is,

$$\bigl\langle \nabla f(x) - \nabla f(y), x - y \bigr\rangle \ge \frac{1}{L} \bigl\| \nabla f(x) - \nabla f(y) \bigr\|^2, \quad \forall x, y \in C.$$

Repeating the same arguments as in the proof of Proposition 3.1, we know that $\nabla f_\alpha = \alpha I + \nabla f$ is $1/(\alpha + L)$-ism. It is clear that C_n is closed and Q_n is closed and convex for every $n = 1, 2, \ldots$. As the defining inequality in C_n is equivalent to the inequality

$$\bigl\langle 2(x_n - z_n), z \bigr\rangle \le \|x_n\|^2 - \|z_n\|^2 + \theta_n,$$

by Lemma 2.5 we also have that C_n is convex for every $n = 1, 2, \ldots$. As $Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}$, we have $\langle x_n - z, x - x_n \rangle \ge 0$ for all $z \in Q_n$, and by Proposition 2.1(i) we get $x_n = P_{Q_n}x$.

We divide the rest of the proof into several steps.

Step 1. $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.

Indeed, take $u \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$ arbitrarily. Taking into account $\lambda_n(\alpha_n + L) < 1$, $\forall n \ge 1$ and repeating the same arguments as in (3.5) and (3.6), we deduce that

$$\|y_n - u\| \le \frac{1}{\mu_n} \bigl(\|x_n - u\| + \lambda_n \alpha_n \|u\| \bigr) \tag{4.2}$$

and

$$\|t_n - u\| \le \frac{1 + 2\mu_n}{\mu_n^2} \bigl(\|x_n - u\| + \lambda_n \alpha_n \|u\| \bigr). \tag{4.3}$$

Thus, from (4.2) and (4.3) it follows that

$$\|y_n - u\| + (1 - \mu_n)\|t_n - u\| \le \frac{3}{\mu_n^2} \bigl(\|x_n - u\| + \lambda_n \alpha_n \|u\| \bigr),$$

which together with $\lambda_n(\alpha_n + L) < 1$ implies that

$$\begin{aligned} \bigl[\|y_n - u\| + (1 - \mu_n)\|t_n - u\| \bigr]^2 &\le \frac{9}{\mu_n^4} \bigl(\|x_n - u\| + \lambda_n \alpha_n \|u\| \bigr)^2 \\ &\le \frac{9}{\mu_n^4} \bigl(2\|x_n - u\|^2 + 2\|u\|^2 \bigr) \\ &= \frac{18}{\mu_n^4} \|x_n - u\|^2 + \frac{18}{\mu_n^4} \|u\|^2. \end{aligned}$$

Repeating the same arguments as in (3.8) and (3.9), we can deduce that

$$\begin{aligned}
&\|t_n - u\|^2 \\
&\quad \le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n(\alpha_n + L)\|x_n - y_n\|\|t_n - y_n\| \\
&\qquad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr] \\
&\quad \le \|x_n - u\|^2 + \bigl(\lambda_n^2(\alpha_n + L)^2 - 1\bigr)\|x_n - y_n\|^2 + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr] \\
&\quad \le \|x_n - u\|^2 + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr]
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\|z_n - u\|^2 &\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|t_n - u\|^2 \\
&\le \|x_n - u\|^2 + (1-\gamma_n)\bigl(\lambda_n^2(\alpha_n + L)^2 - 1\bigr)\|x_n - y_n\|^2 \\
&\quad + 2\lambda_n\alpha_n\|u\|\bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr] \\
&\le \|x_n - u\|^2 + \alpha_n\bigl[\lambda_n^2\|u\|^2 + \bigl(\|y_n - u\| + (1-\mu_n)\|t_n - u\|\bigr)^2\bigr] \\
&\le \|x_n - u\|^2 + \alpha_n\frac{18}{\mu_n^4}\left[\|x_n - u\|^2 + \left(1 + \frac{1}{18L^2}\right)\|u\|^2\right] \\
&\le \|x_n - u\|^2 + \alpha_n\frac{18}{\mu_n^4}\Delta_n \\
&= \|x_n - u\|^2 + \theta_n
\end{aligned} \tag{4.5}$$

for every $n = 1, 2, \dots$ and hence $u \in C_n$. So,

$$\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_n, \quad \forall n \ge 1.$$

Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_n \cap Q_n$ for every $n = 1, 2, \dots$. For $n = 1$ we have $Q_1 = C$. Hence we obtain $\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_1 \cap Q_1$. Suppose that x_k is given and $\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_k \cap Q_k$ for some integer $k \ge 1$. Since $\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma$ is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of C. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in C_k \cap Q_k$. Since $\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in \bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma$ and hence $\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset Q_{k+1}$. Therefore, we obtain $\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_{k+1} \cap Q_{k+1}$.

Step 2. $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Indeed, let $q = P_{\bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n)\cap\Gamma}x$. From $x_{n+1} = P_{C_n \cap Q_n}x$ and $q \in \bigcap_{n=1}^{\infty}\operatorname{Fix}(S_n) \cap \Gamma \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|q - x\| \tag{4.6}$$

for every $n = 1, 2, \dots$. Therefore, $\{x_n\}$ is bounded. From (4.2), (4.3) and (4.5) we also obtain that $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}x$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\|$$

for every $n = 1, 2, \ldots$. Therefore, there exists $\lim_{n\to\infty} \|x_n - x\|$. Since $x_n = P_{Q_n}x$ and $x_{n+1} \in Q_n$, using Proposition 2.1(ii), we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n,$$

which implies that

$$\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n}.$$

Hence we get

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_{n+1} - x_n\| + \sqrt{\theta_n}$$

for every $n = 1, 2, \ldots$. From $\|x_{n+1} - x_n\| \to 0$ and $\theta_n \to 0$, we conclude that $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Step 3. $\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - t_n\| = 0$ and $\lim_{n\to\infty} \|x_n - W_n x_n\| = \lim_{n\to\infty} \|x_n - Wx_n\| = 0$.

Indeed, since $\{\gamma_n\} \subset [0, c]$, $\{\lambda_n\} \subset [a, b]$, $1 - b^2L^2 > 0$, $\alpha_n \to 0$ and $\|x_n - z_n\| \to 0$ as $n \to \infty$, from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ we conclude from (4.5) that

$$\begin{aligned}
&\|x_n - y_n\|^2 \\
&\quad \le \frac{1}{(1-\gamma_n)(1-\lambda_n^2(\alpha_n + L)^2)} \bigl\{ \|x_n - u\|^2 - \|z_n - u\|^2 \\
&\qquad + 2\lambda_n\alpha_n\|u\| \bigl[\|y_n - u\| + (1-\mu_n)\|t_n - u\| \bigr] \bigr\} \\
&\quad \le \frac{1}{(1-\gamma_n)(1-\lambda_n^2(\alpha_n + L)^2)} \bigl\{ \bigl(\|x_n - u\| + \|z_n - u\| \bigr) \|x_n - z_n\| \\
&\qquad + 2\lambda_n\alpha_n\|u\| \bigl[\|y_n - u\| + \|t_n - u\| \bigr] \bigr\} \\
&\quad \le \frac{1}{(1-c)(1-b^2(\alpha_n + L)^2)} \bigl\{ \bigl(\|x_n - u\| + \|z_n - u\| \bigr) \|x_n - z_n\| \\
&\qquad + 2\lambda_n\alpha_n\|u\| \bigl[\|y_n - u\| + \|t_n - u\| \bigr] \bigr\} \\
&\quad \to 0.
\end{aligned}$$

Utilizing the arguments similar to those in (4.4),

$$\begin{aligned}
&\|t_n - u\|^2 \\
&\quad \le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n(\alpha_n + L)\|x_n - y_n\|\|t_n - y_n\|
\end{aligned}$$

$$\begin{aligned}&+2\lambda_n\alpha_n\|u\|\bigl[\|y_n-u\|+(1-\mu_n)\|t_n-u\|\bigr]\\ &\le\|x_n-u\|^2+\bigl(\lambda_n^2(\alpha_n+L)^2-1\bigr)\|t_n-y_n\|^2+2\lambda_n\alpha_n\|u\|\bigl[\|y_n-u\|+(1-\mu_n)\|t_n-u\|\bigr].\end{aligned}$$

Hence,

$$\begin{aligned}\|z_n-u\|^2&\le\gamma_n\|x_n-u\|^2+(1-\gamma_n)\|t_n-u\|^2\\ &\le\gamma_n\|x_n-u\|^2+(1-\gamma_n)\bigl\{\|x_n-u\|^2+\bigl(\lambda_n^2(\alpha_n+L)^2-1\bigr)\|t_n-y_n\|^2\\ &\quad+2\lambda_n\alpha_n\|u\|\bigl[\|y_n-u\|+(1-\mu_n)\|t_n-u\|\bigr]\bigr\}\\ &\le\|x_n-u\|^2+(1-\gamma_n)\bigl(\lambda_n^2(\alpha_n+L)^2-1\bigr)\|t_n-y_n\|^2\\ &\quad+2\lambda_n\alpha_n\|u\|\bigl[\|y_n-u\|+\|t_n-u\|\bigr].\end{aligned}$$

Since $\{\gamma_n\}\subset[0,c]$, $\{\lambda_n\}\subset[a,b]$, $1-b^2L^2>0$, $\alpha_n\to0$ and $\|x_n-z_n\|\to0$ as $n\to\infty$, from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ we deduce that

$$\begin{aligned}&\|t_n-y_n\|^2\\ &\quad\le\frac{1}{(1-\gamma_n)(1-\lambda_n^2(\alpha_n+L)^2)}\bigl\{\|x_n-u\|^2-\|z_n-u\|^2\\ &\qquad+2\lambda_n\alpha_n\|u\|\bigl[\|y_n-u\|+\|t_n-u\|\bigr]\bigr\}\\ &\quad\le\frac{1}{(1-\gamma_n)(1-\lambda_n^2(\alpha_n+L)^2)}\bigl\{\bigl(\|x_n-u\|+\|z_n-u\|\bigr)\|x_n-z_n\|\\ &\qquad+2\lambda_n\alpha_n\|u\|\bigl[\|y_n-u\|+\|t_n-u\|\bigr]\bigr\}\\ &\quad\le\frac{1}{(1-c)(1-b^2(\alpha_n+L)^2)}\bigl\{\bigl(\|x_n-u\|+\|z_n-u\|\bigr)\|x_n-z_n\|\\ &\qquad+2\lambda_n\alpha_n\|u\|\bigl[\|y_n-u\|+\|t_n-u\|\bigr]\bigr\}\\ &\quad\to0.\end{aligned}$$

Taking into consideration that

$$\|x_n-t_n\|\le\|x_n-y_n\|+\|y_n-t_n\|,$$

we also have

$$\lim_{n\to\infty}\|x_n-t_n\|=0.$$

Since $(1-\gamma_n)(W_nt_n-t_n)=\gamma_n(t_n-x_n)+z_n-t_n$, we get

$$\begin{aligned}(1-c)\|W_nt_n-t_n\|&\le(1-\gamma_n)\|W_nt_n-t_n\|\\ &\le\gamma_n\|t_n-x_n\|+\|z_n-t_n\|\\ &\le(1+\gamma_n)\|t_n-x_n\|+\|z_n-x_n\|,\end{aligned}$$

and hence $\|t_n - W_n t_n\| \to 0$. Observe also that

$$\begin{aligned}\|x_n - W_n x_n\| &\le \|x_n - t_n\| + \|t_n - W_n t_n\| + \|W_n t_n - W_n x_n\| \\ &\le 2\|x_n - t_n\| + \|t_n - W_n t_n\|.\end{aligned}$$

So, we have $\|x_n - W_n x_n\| \to 0$. On the other hand, since $\{x_n\}$ is bounded, from Lemma 2.8, we have $\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0$. Therefore, we obtain

$$\lim_{n\to\infty} \|x_n - W x_n\| = 0.$$

Step 4. $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$.

Indeed, repeating the same arguments as in the proof of Proposition 3.4, we can derive the desired conclusion.

Step 5. $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{\mathrm{Fix}(S)\cap\Gamma} x$.

Indeed, take $\hat{u} \in \omega_w(x_n)$ arbitrarily. Then $\hat{u} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$ according to Step 4. Moreover, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{u}$. Hence, from $q = P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)\cap\Gamma} x$, $\hat{u} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$, and (4.6), we have

$$\|q - x\| \le \|\hat{u} - x\| \le \liminf_{i\to\infty} \|x_{n_i} - x\| \le \limsup_{i\to\infty} \|x_{n_i} - x\| \le \|q - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|\hat{u} - x\|.$$

From $x_{n_i} - x \rightharpoonup \hat{u} - x$ we have $x_{n_i} - x \to \hat{u} - x$ due to the Kadec-Klee property of Hilbert spaces [36]. So, it is clear that $x_{n_i} \to \hat{u}$. Since $x_n = P_{Q_n} x$ and $q \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|q - x_{n_i}\|^2 = \langle q - x_{n_i}, x_{n_i} - x\rangle + \langle q - x_{n_i}, x - q\rangle \ge \langle q - x_{n_i}, x - q\rangle.$$

As $i \to \infty$, we obtain $-\|q - \hat{u}\|^2 \ge \langle q - \hat{u}, x - q\rangle \ge 0$ by $q = P_{\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)\cap\Gamma} x$ and $\hat{u} \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \Gamma$. Hence we have $\hat{u} = q$. This implies that $x_n \to q$. It is easy to see that $y_n \to q$ and $z_n \to q$. This completes the proof. □

Remark 4.1 Our Theorem 4.1 improves, extends, supplements, and develops Nadezhkina and Takahashi [9, Theorem 3.1] and Ceng *et al.* [11, Theorem 3.1] in the following aspects.

(i) The combination of the problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in [9, Theorem 3.1] and the one of finding an element of $\bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A)$ in [11, Theorem 3.1] is extended to develop the one of finding an element of $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap \Gamma$ in our Theorem 4.1.

(ii) Our Theorem 3.1 is one strong convergence result and drops the required condition $\liminf_{n\to\infty} \langle Ax_n, x - x_n\rangle \ge 0$, $\forall x \in C$ in [11, Theorem 3.1].

(iii) The iterative scheme in [9, Theorem 3.1] is extended to develop the iterative scheme (4.1) of our Theorem 4.1 by virtue of the iterative scheme of [11, Theorem 3.1]. The iterative scheme (4.1) of our Theorem 4.1 is more advantageous and more flexible than the iterative scheme of [9, Theorem 3.1] because it involves several parameter sequences $\{\lambda_n\}$, $\{\mu_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$.

(iv) The iterative scheme (4.1) in our Theorem 4.1 is very different from every one in [9, Theorem 3.1] and [11, Theorem 3.1] because two explicit iteration steps of computing y_n and z_n in [9, Theorem 3.1] and [11, Theorem 3.1] are replaced by three iteration steps involving two implicit steps in the iterative scheme (3.1) of our Theorem 3.1.

(v) The argument technique of our Theorem 4.1 combines the argument one in [9, Theorem 3.1] and the argument one in [11, Theorem 3.1]. Because the problem of finding an element of $\bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i) \cap \Gamma$ in our Theorem 4.1 involves a countable family of nonexpansive mappings $\{S_n\}$, the proof of our Theorem 4.1 depends on the properties of the W-mapping (see Lemmas 2.6-2.8 of Section 2 in this paper). Therefore, the proof of our Theorem 4.1 is very different from every one in [9, Theorem 3.1] and [11, Theorem 3.1].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contribute equally to this work. All authors read and approved the final manuscript.

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